

# KNOT FLOER HOMOLOGY AND KHOVANOV-ROZANSKY HOMOLOGY FOR SINGULAR LINKS

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**ABSTRACT.** The untwisted cube of resolutions for knot Floer homology assigns a chain complex  $C(S)$  to each singular resolution  $S$  of a knot  $K$ . It was conjectured by Manolescu that the homology of this complex is isomorphic to the HOMFLY-PT homology of  $S$  defined by Khovanov and Rozansky. We show that, like HOMFLY-PT homology, the homology of  $C(S)$  has a family of spectral sequences  $E_k(n)$  for  $n \geq 2$  converging to the  $sl_n$  homology of  $S$ .

## 1. INTRODUCTION

The past two decades have seen tremendous growth within the field of knot theory, with the most powerful tools coming from knot homology theories. Of these homology theories there are two main types: those with roots in representation theory, such as HOMFLY-PT and  $sl_n$  homologies of Khovanov and Rozansky, and those constructed via symplectic geometry, such as the knot Floer homology of Ozsváth-Szabó and Rasmussen, also known as *HFK*. Despite the fundamental differences between these types of theories, there seem to be deep connections between them, such as the following conjecture:

**Conjecture 1.1** ([2]). *For any knot  $K$ , there is a spectral sequence whose  $E_2$  page is the HOMFLY-PT homology of  $K$  and whose  $E_\infty$  page is the knot Floer homology of  $K$ .*

Rasmussen constructed a family of spectral sequences  $E_k(n)$  for  $n \geq 1$  whose  $E_2$  page is HOMFLY-PT homology and whose  $E_\infty$  page is  $sl_n$  homology. The above conjecture would fit into this framework as an  $n = 0$  spectral sequence.

Since the HOMFLY-PT complex is constructed via an oriented cube of resolutions, it is natural to consider a similar construction for knot Floer homology. The oriented cube of resolutions for *HFK* was first constructed by Ozsváth and Szabó with twisted coefficients. They noted similarities between their construction and HOMFLY-PT homology, which were studied extensively by Gilmore [3].

The cube of resolutions construction was modified by Manolescu to give an untwisted complex, which will be the focal point of our discussion. In this paper, Manolescu proposed the following:

**Conjecture 1.2** ([8]). *Let  $S$  be a singular resolution of a knot  $K$ , and let  $C(S)$  denote the chain complex that the knot Floer cube of resolutions assigns to the vertex corresponding to  $S$ . Then*

$$H(C(S)) \cong H^{KR}(S)$$

where  $H^{KR}(S)$  is the HOMFLY-PT homology of  $S$ .

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Together with cooperation from the edge maps, this conjecture would show that the spectral sequence in Conjecture 1.1 is induced by the cube filtration on the knot Floer cube of resolutions. This paper provides evidence for Conjecture 1.2.

Rasmussen's spectral sequences  $E_k(n)$  exist both on the level of knots and on the level of singular resolutions - the only difference is that for singular links the  $E_1$  page is HOMFLY-PT homology rather than the  $E_2$  page, because we don't have edge maps to worry about. In particular, there are spectral sequences  $E_k(n)$  such that  $E_1(n) \cong H^{KR}(S)$  and  $E_\infty \cong H_n^{KR}(S)$ , where  $H_n^{KR}(S)$  denotes the  $sl_n$  homology of  $S$ . We show that for  $n \geq 2$ , the same holds true for  $H(C(S))$ .

**Theorem 1.3.** *For all  $n \geq 2$ , there is a spectral sequence whose  $E_1$  page is  $H(C(S))$  and whose  $E_\infty$  page is  $H_n^{KR}(S)$ .*

For knots, the  $sl_n$  homology  $H_n^{KR}(S)$  stabilizes to  $H(C(S))$  (the reduced versions) for  $n$  sufficiently large, so one could use a limiting argument together with Theorem 1.3 to prove Conjecture 1.2. Unfortunately, the HOMFLY-PT homology for singular diagrams is infinite dimensional, so more is needed in this case.

We finish the introduction with a brief description of the spectral sequences. First, we filter the complex  $C(S)$  by adding additional basepoints to the Heegaard diagram for  $S$ . The filtered homology decomposes as a direct sum

$$H(C(S), d_0) \cong \bigoplus_Z H^{KR}(S - Z)$$

where  $Z$  is a multi-cycle in  $S$ , i.e. a subset of the edges of  $S$  which make a disjoint union of oriented circles in the plane, and  $S - Z$  is the diagram obtained by removing the edges in  $Z$  from  $S$ .

We then add in additional differentials to  $C(S)$  by counting extra holomorphic discs, creating a complex  $C_n(S)$ . This complex locally resembles a matrix factorization with the same potential as the  $sl_n$  complex. Utilizing the same basepoint filtration as above, we show that the filtered homology of this new complex is given by

$$H(C_n(S), d) \cong \bigoplus_Z H_n^{KR}(S - Z)$$

By a grading argument, the filtered homology is actually the same as the total homology  $H(C_n(S), d_0)$ , i.e. the spectral sequence collapses at the  $E_1$  page. Moreover, the composition product formula of Wagner says that

$$\bigoplus_Z H_n^{KR}(S - Z) \cong H_{n+1}^{KR}(S)$$

Putting this all together, we see that

$$H(C_n(S), d) \cong H_{n+1}^{KR}(S)$$

The final step of the proof is showing that the differentials all decrease the Alexander grading while the old differentials preserve it, so the spectral sequence induced by the Alexander filtration has  $E_1$  page  $H(C(S))$  and  $E_\infty$  page  $H_{n+1}^{KR}(S)$ .

## 2. SINGULAR RESOLUTIONS AND THE GROUND RING

A complete resolution  $S$  of a knot  $K$  in braid position can be viewed as an oriented planar graph with the following properties:

1. All vertices are either 2-valent or 4-valent.
2. The number of incoming edges is equal to the number of outgoing edges at each vertex.
3. If  $Z$  is an oriented cycle in  $S$ , then the unique disc  $D \subset \mathbb{R}^2$  with boundary  $Z$  intersects the center of the braid.

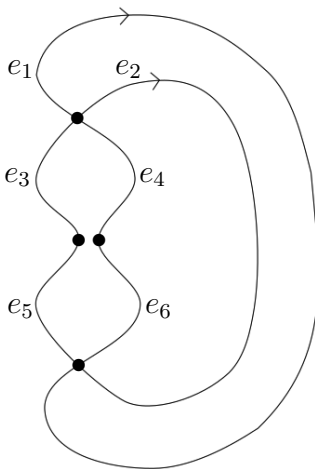


FIGURE 1. An example of a singular braid diagram

Let  $e_1, \dots, e_n$  denote the edges of  $S$ . To each edge  $e_i$ , we assign an indeterminant  $U_i$ . All three homology theories will be defined over the ground ring  $R = \mathbb{Q}[U_1, \dots, U_n]$ .

## 3. HOMFLY-PT HOMOLOGY AND $sl_n$ HOMOLOGY

This section will give a brief description of the HOMFLY-PT and  $sl_n$  as defined in [6] and [7]. We will use the grading conventions from [11], though we will leave out the overall grading shifts coming from the braid number. The reader can refer to these resources for further background. The HOMFLY-PT and  $sl_n$  complexes have the same generators, with the  $sl_n$  complex having strictly more differentials than the HOMFLY-PT complex. For this reason, we will start by defining the HOMFLY-PT complex, then we will describe the additional differentials to make the  $sl_n$  complex.

The HOMFLY-PT complex for links comes equipped with a triple-grading, and the  $sl_n$  complex with a bigrading. One of the gradings in both theories, however, comes from the height in the cube of resolutions, so it will be fixed for a single resolution. The HOMFLY-PT complex will therefore come with a bigrading, and the  $sl_n$  complex with a single grading. For the HOMFLY-PT complex, these gradings are called the quantum grading, denoted  $gr_q$ , and the horizontal grading, denoted  $gr_h$ . Multiplication by the  $U_i$  increases the quantum grading by 2 and preserves the horizontal grading.

Let  $V_2(S)$  denote the 2-valent vertices in  $S$  and  $V_4(S)$  the 4-valent vertices of  $S$ . For vertices  $v$  in  $V_2(S)$ , there is a unique outgoing edge  $e_i$  and a unique incoming edge  $e_j$ . Define  $L(v)$  to be the linear term  $U_i - U_j$ . Similarly, for vertices  $v$  in  $V_4(S)$  there are two outgoing edges  $e_i$  and  $e_j$  and two incoming edges  $e_k$  and  $e_l$ . We define  $L(v)$  to be the linear term  $U_i + U_j - U_k - U_l$  and  $Q(v)$  to be the quadratic term  $U_i U_j - U_k U_l$ .

The HOMFLY-PT complex is a tensor product of complexes  $C^{KR}(v)$  for each vertex  $v$ . For  $v$  in  $V_2(S)$ ,  $C^{KR}(v)$  is defined as

$$R\{0, -2\} \xrightarrow{L(v)} R\{0, 0\}$$

where  $R\{i, j\}$  refers to the thing  $R$  shifted by  $i$  in  $gr_q$  and by  $j$  in  $gr_h$ . For  $v$  in  $V_4(S)$ ,  $C^{KR}(v)$  is defined as

$$\begin{array}{ccc} R\{1, -4\} & \xrightarrow{L(v)} & R\{1, -2\} \\ \downarrow Q(v) & & \downarrow Q(v) \\ R\{-1, -2\} & \xrightarrow{L(v)} & R\{-1, 0\} \end{array}$$

Note that the differential is homogeneous of degree  $\{2, 2\}$ . The HOMFLY-PT complex for the singular diagram  $S$  is given by  $C^{KR}(S)$

$$C^{KR}(S) = \bigotimes_{v \in S} C^{KR}(v)$$

and the HOMFLY-PT homology  $H^{KR}(S)$  is the homology of  $C^{KR}(S)$ .

We will now define the additional differentials which give  $sl_n$  homology. For a vertex  $v$  in  $S$  with outgoing edges  $E_{out}$  and incoming edges  $E_{in}$ , let the potential  $w_n$  be given by

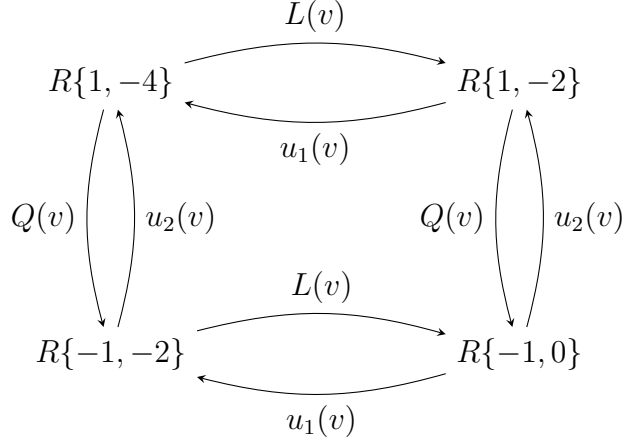
$$w_n(v) = \sum_{e_i \in E_{out}} U_i^{n+1} - \sum_{e_j \in E_{in}} U_j^{n+1}$$

For  $v$  in  $V_2(S)$ , let  $u_1(v)$  be the unique element in  $R$  such that  $u_1(v)L(v) = w_n(v)$ . For  $v$  in  $V_4(S)$ , we can choose  $u_1(v)$  and  $u_2(v)$  such that  $u_1(v)L(v) + u_2(v)Q(v) = w_n(v)$ . Unlike the 2-valent case, the choice is not unique, but the reader can refer to ([?], p. 6) for the precise choice. (It is not relevant for our discussion.)

For each vertex  $v$ , we will add new differentials to  $C^{KR}(v)$  to make a new complex  $C_n^{KR}(V)$ . For  $v$  in  $V_2(S)$ ,  $C_n^{KR}(v)$  is given by

$$\begin{array}{ccc} & \xrightarrow{L(v)} & \\ R\{0, -2\} & \xrightarrow{\quad} & R\{0, 0\} \\ & \xleftarrow{u_1(v)} & \end{array}$$

and for  $v$  in  $V_4(S)$ ,  $C_n^{KR}(v)$  is given by



Observe that for both types of vertices, the differential on  $C_n(v)$  satisfies  $d^2 = w_n(v)I$ . Such a complex is called a matrix factorization with potential  $w_n$ . Since  $d^2$  is non-zero, its homology is not well-defined. However, we are interested in the tensor product of  $C_n(v)$  over all vertices  $v$  in  $S$ . Define the  $sl_n$  complex  $C_n(S)$  by

$$C_n(S) = \bigotimes_{v \in S} C_n(v)$$

As mentioned above, the HOMFLY-PT differentials are homogeneous of degree  $\{2, 2\}$ . These differentials are denoted by  $d_+$ . The new differentials, those with coefficients  $u_1(v)$  and  $u_2(v)$ , are homogeneous of degree  $\{2n, -2\}$ . These are denoted  $d_-$ . The total differential  $d_{tot} = d_+ + d_-$  is not homogeneous in this bigrading. However, if we look at the grading  $gr_n = gr_q + (n-1)gr_h/2$ , then  $d_{tot}$  is homogeneous of degree  $n+1$ .

Additionally,  $d_{tot}^2 = 0$ . This can be seen from the fact that the potential is additive under tensor product, so  $d_{tot}^2 = \sum_{v \in S} w_n(v)$ . The sum must be zero because each edge  $e_i$  is an outgoing edge for one vertex, which will contribute  $U_i^{n+1}$ , and an incoming edge for another vertex, which will contribute  $-U_i^{n+1}$ .

We have shown that  $C_n(S)$  is a well-defined chain complex which is homogeneous with respect to the grading  $gr_n$ . We define the  $sl_n$  homology  $H_n(S)$  to be the homology of this complex.

*Remark 3.1.* The definitions given here correspond to the unreduced theories in [11] as opposed to the middle homologies. This choice will make our arguments involving the composition product easier, although they would still work for the middle versions using the destabilized composition product in [1].

**3.1. Rasmussen's Spectral Sequences.** Rasmussen showed in [11] that there are a family of spectral sequences  $E_k(n)$  which start at HOMFLY-PT homology and converge to  $sl_n$  homology. These spectral sequences are somewhat difficult to prove for the case of knots and links, but they are much simpler for fully singular diagrams.

The differential  $d_{tot}$  on  $C_n(S)$  admits a filtration induced by the grading  $(gr_q - gr_h)/(2n+2)$ . With respect to this grading,  $d_+$  is homogeneous of degree 0, and  $d_-$  is homogeneous of degree 1. The  $E_1$  page of this spectral sequence is  $H(C_n^{KR}(S), d_+)$ , which is exactly the definition of HOMFLY-PT homology. The  $E_\infty$  page is the homology with respect to  $d_{tot}$ , or  $sl_n$  homology.

It turns out that this spectral sequence collapses at the  $E_2$  page  $H_*(H_*(C_n(S), d_+), d_-^*)$ . We will denote this page by  $H^\pm(C_n^{KR}(S))$ . Note that  $H^\pm(C_n^{KR}(S))$  is bigraded, as both  $d_+$  and  $d_-$  are homogeneous. The fact that all higher differentials are trivial follows from the following lemma.

**Lemma 3.2** ([11]). *The homology  $H^\pm(C_n^{KR}(S))$  lies in a single horizontal grading, namely  $gr_h = -2r(S)$ , where  $r(S)$  is the number of strands in the singular braid  $S$ .*

Since none of the higher differentials preserve the horizontal grading, they must all be trivial, causing the spectral sequence to collapse.

**Corollary 3.3.** *Viewing  $H^\pm(C_n^{KR}(S))$  as singly graded with grading  $gr_n$ , there is a graded isomorphism  $H^\pm(C_n^{KR}(S)) \cong H_n^{KR}(S)$ .*

*Remark 3.4.* The reader familiar with [11] may note that our homology lies in  $gr_h = -2r(S)$ , while Rasmussen's lies in  $gr_h = 1 - r(S)$ . This difference comes from the fact that our homology is unreduced, which changes the grading by 2, and because we are leaving out the overall grading shift of  $r(S) - 1$ .

#### 4. KNOT FLOER HOMOLOGY

We will assume that the reader is familiar with Heegaard diagrams and knot Floer homology. For background on the subject, refer to [9]. The oriented cube of resolutions for  $HFK$  was originally defined with twisted coefficients ([10]), but we will be dealing with the untwisted version defined by Manolescu in [8]. The oriented cube of resolutions uses the Heegaard diagram shown in Figure 2.

Note that unlike in [8], we do not have a marked edge at which an  $\alpha$  and a  $\beta$  circle are removed. Instead, we place an additional  $X$  and  $O$  outside of our braid - we will denote this special  $X$  by  $X_0$ . Since discs are not allowed to pass through  $X_0$ , this can be viewed as puncturing the sphere, making our diagram a truly planar diagram. We will also set the  $U$  corresponding to the  $O$  equal to zero to avoid increasing the ground ring. The net effect of this change is that we have added an unlinked component and reduced it, so the homology is twice that of [8].

The knot Floer complex corresponding to this diagram is  $CFK^-(S)$ . The complex ascribed to a vertex in the cube of resolutions, which we will denote  $C(S)$ , is the tensor product of  $CFK^-(S)$  with a certain Koszul complex. Using the terminology from the previous section, we can define  $C(S)$  as follows:

$$C(S) = CFK^-(S) \otimes \bigotimes_{v \in V_4(S)} R \xrightarrow{L(v)} R$$

We will denote the Koszul complex by  $K(S)$ .

**4.1. The Generators of  $CFK^-(S)$ .** In order to understand the homology of  $C(S)$ , we are going to need some tools for understanding  $CFK^-(S)$ . Let  $E(S)$  denote the set of edges of  $S$ , and let  $x$  be a generator of the complex  $CFK^-(S)$  (i.e. an  $n$ -tuple of intersection points of the  $\alpha$  and  $\beta$  curves). We ascribe a subset  $Z$  of  $E$  to the generator  $x$  as follows.

Each  $O_i$  in the Heegaard diagram is contained in a unique minimal bigon. The boundary of this bigon contains two intersection points - if either of these intersection points are in the  $n$ -tuple  $x$ , then  $e_i$  is in  $Z$ . For example, in Figure 2a, there are 5 types of generators:  $(a, d)$ ,

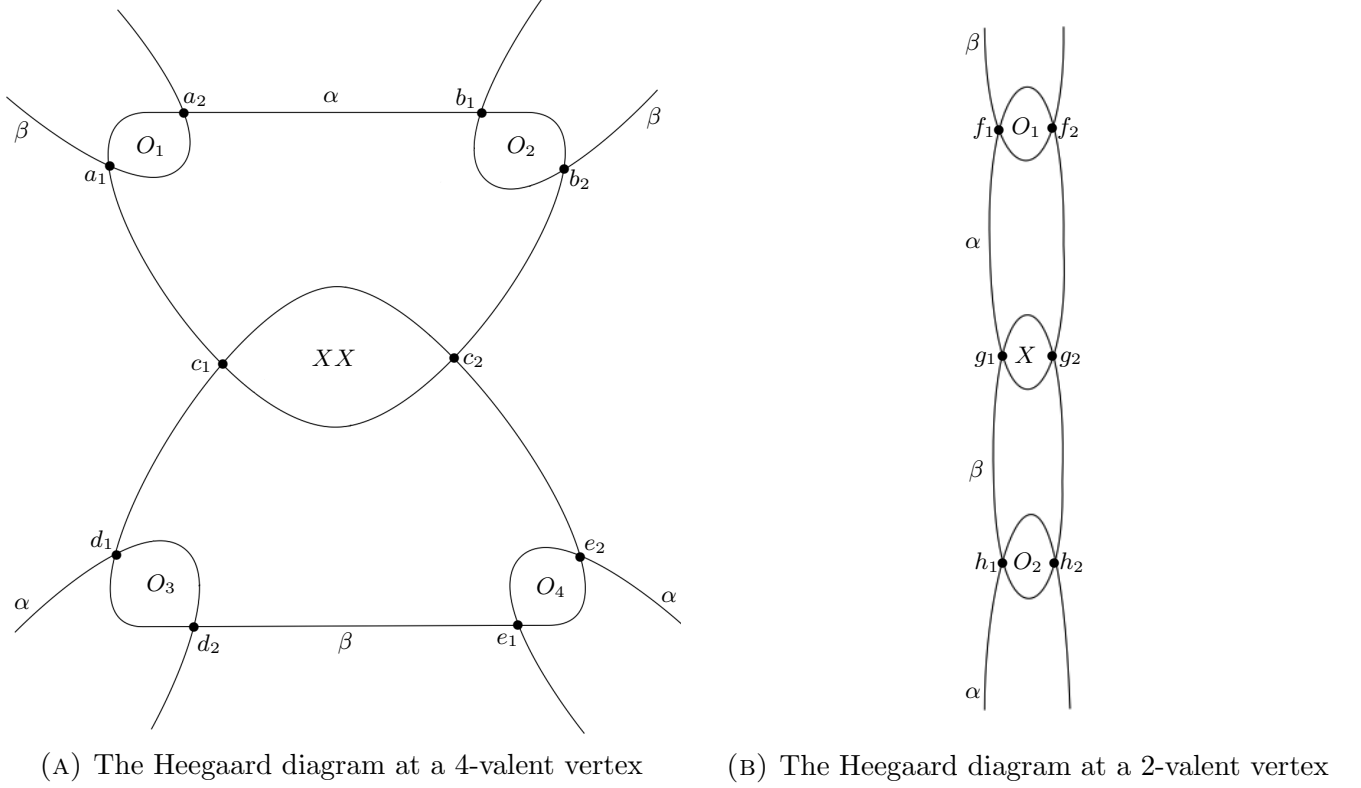


FIGURE 2. The local Heegaard diagram for a singular link

$(a, e)$ ,  $(b, d)$ ,  $(b, e)$ , and  $(c)$ . The underlying (local) cycles of these generators are  $e_1e_3$ ,  $e_1e_4$ ,  $e_2e_3$ ,  $e_2e_4$ , and  $\emptyset$ , respectively.

As observed in [10],  $Z$  must satisfy two conditions. First, for any vertex  $v$  in  $S$ , the number of incoming edges in  $Z$  must equal the number of outgoing edges in  $Z$ , and second,  $Z$  can not contain all four edges at any 4-valent vertex in  $S$ . In other words,  $Z$  must be a disjoint union of oriented circles contained in  $S$ . We call such a set of edges a *multi-cycle*.

Let  $CFK^-(Z)$  denote the  $R$ -module spanned by generators  $x$  where the multi-cycle underlying  $x$  is  $Z$ , and let  $C(Z)$  be the tensor product of  $CFK^-(Z)$  with the Koszul complex.

$$C(Z) = CFK^-(Z) \otimes \bigotimes_{v \in V_4(S)} R \xrightarrow{L(v)} R$$

**4.2. A Filtration on  $CFK^-(S)$ .** It turns out that there is a filtration  $F$  on  $CFK^-(S)$  that divides generators according to their underlying cycles. In other words, if there is a filtration-preserving differential  $d$  with  $d(x) = y$ , then  $x$  and  $y$  have the same underlying cycle.

This filtration is induced by placing additional basepoints  $p_i$  in our Heegaard diagram as shown in Figure 3. The markings  $p_i$  are in canonical bijection with regions in  $\mathbb{R}^2 - S$ .

**Lemma 4.1.** *These markings define a filtration on the complex  $CFK^-(S)$ , where the change in filtration level of a differential is given by the sum of the multiplicities of the corresponding holomorphic disc at these basepoints.*

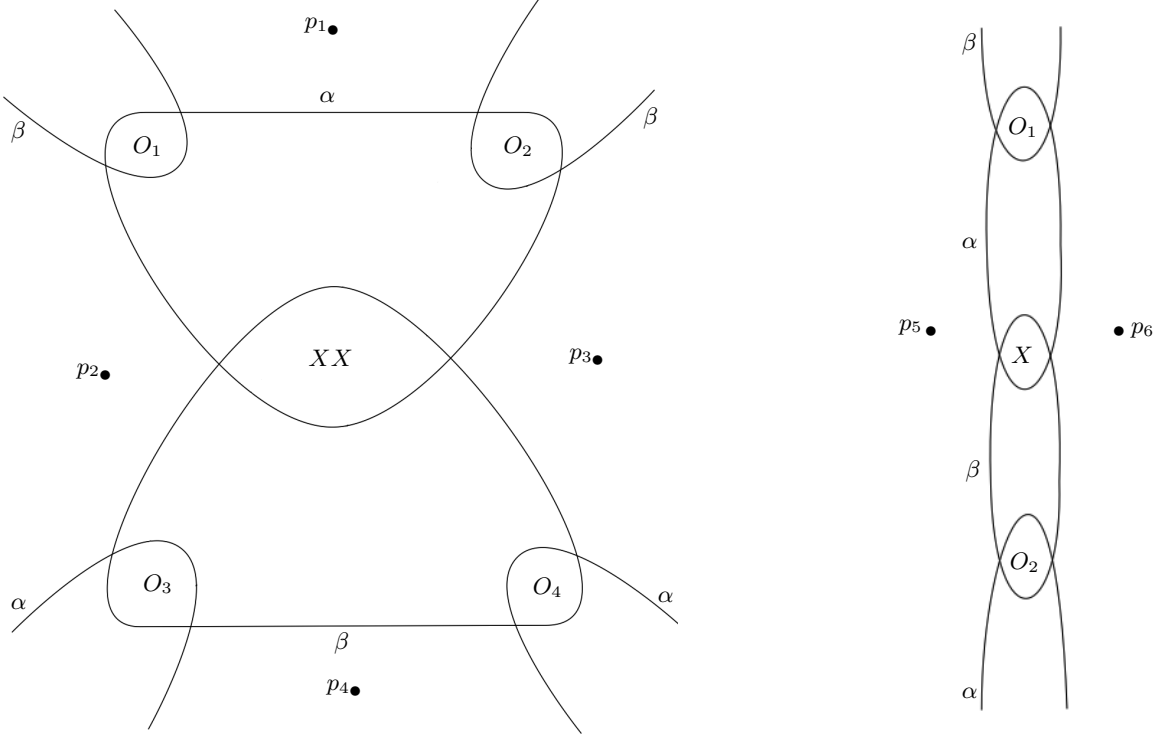


FIGURE 3. Local Diagrams with Additional Markings

*Proof.* It is sufficient to show that any periodic domain has multiplicity zero at these markings. This follows from that fact that for any  $\alpha$  or  $\beta$  circle, the markings and  $X_0$  lie on the same side. So for any periodic domain, the multiplicity at any of these points is the same as that of  $X_0$ , which is required to be zero.

□

We extend this filtration to  $C(S)$  by placing the Koszul complex in a single filtration level. Let  $d_k$  denote the component of the differential on  $C(S)$  which increases the filtration by  $k$ .

**Lemma 4.2.** *The differential  $d_0$  preserves  $C(Z)$ , i.e. it does not change the underlying cycle of a generator.*

*Proof.* Each basepoint gives a filtration on our complex corresponding to a region in the knot projection. Let  $x$  be a generator with multi-cycle  $Z$ , and let  $C$  be an oriented 2-chain with boundary  $Z$ . If we require that  $C$  has multiplicity 0 on the outer region (the one corresponding to  $X_0$ ), it is clear that this 2-chain is unique.

Within the planar Heegaard diagram for  $K$ , we can find a disc that connects  $x$  to a generator corresponding to the empty cycle (see [10], section 3), and since each region in the knot projection contains a basepoint, the multiplicities of the disc at each basepoint will be equal to the multiplicity of  $C$  in that region. Thus, each filtration level uniquely determines a 2-chain  $C$ , whose boundary gives the multi-cycle  $Z$ . Since no two multi-cycles correspond to the same 2-chain, this completes the proof.

□

This means that the filtered homology, also called the homology of the associated graded object, must split over the multi-cycles  $Z$ .



**4.3. Homology of a Cycle.** Before computing the homology  $H(C(Z), d_0)$ , we will need a definition. If  $S$  is a singular braid and  $Z$  is a multi-cycle in  $S$ , let  $S - Z$  denote the diagram obtained by removing all edges in  $Z$  from  $S$ . Note that  $S - Z$  is still a singular braid because  $Z$  is an oriented cycle in the graph.

Given a cycle  $Z$ , the complex  $CFK^-(Z)$  is easy to compute. Each intersection point in the Heegaard diagram lies on a unique convex bigon (convex in the traditional planar geometry sense), and this bigon either contains an  $X$ , an  $XX$ , or a  $O_i$ . There are canonical bijections between the  $O_i$  bigons and the edges  $e_i$ , between the  $X$  bigons and  $V_2(S)$  and between the  $XX$  bigons and  $V_4(S)$ .

Given a generator  $x$ , let  $W_2(x)$  denote the set of vertices at which  $x$  has an intersection point on one of the  $X$  bigons, and let  $W_4(x)$  denote the set of vertices at which  $x$  has an intersection point on one of the  $XX$  bigons. Note that  $W_2(x)$  and  $W_4(x)$  are uniquely determined by the underlying cycle  $Z$  of  $x$ . In particular,  $W_2(x)$  and  $W_4(x)$  are those vertices which are not endpoints of any edges in  $Z$ . We can therefore define  $W_2(Z)$  and  $W_4(Z)$  accordingly.

The complex for a cycle  $Z$  can now be described as follows. Each edge  $e_i$  in  $Z$  corresponds to two intersection points, which are connected by a bigon containing  $O_i$ . These are the only filtered differentials involving these two intersection points, so  $CFK^-(Z)$  is going to come with a tensor summand of the Koszul complex

$$\bigotimes_{e_i \in Z} R \xrightarrow{U_i} R$$

Each vertex  $v$  in  $W_2(Z)$  also corresponds to two intersection points. They are connected by two bigons, one which passes through  $O_i$  (where  $e_i$  is the outgoing edge from  $v$ ) and one which passes through  $O_j$  (where  $e_j$  is the incoming edge at  $v$ ). These two bigons will give a coefficient of  $\pm(U_i - U_j)$ . Thus, we also get a tensor summand of the Koszul complex

$$\bigotimes_{v \in W_2(Z)} R \xrightarrow{L(v)} R$$

Proving the signs requires slightly more advance machinery, and will be discussed at the end of the section.

Finally, the vertices  $v$  in  $W_4(Z)$  correspond two two intersection points, also connected by two bigons. One passes through  $O_i$  and  $O_j$ , where  $e_i$  and  $e_j$  are the outgoing edges of  $v$ , and the other passes through  $O_k$  and  $O_l$ , where  $e_k$  and  $e_l$  are the incoming edges at  $v$ . These two bigons will contribute a coefficient of  $\pm(U_i U_j - U_k U_l)$ , giving us the last Koszul complex

$$\bigotimes_{v \in W_4(Z)} R \xrightarrow{Q(v)} R$$

These are all the generators and all the differentials, so the total complex is given by

$$CFK^-(Z) = \left[ \bigotimes_{e_i \in Z} R \xrightarrow{U_i} R \right] \otimes \left[ \bigotimes_{v \in W_2(Z)} R \xrightarrow{L(v)} R \right] \otimes \left[ \bigotimes_{v \in W_4(Z)} R \xrightarrow{Q(v)} R \right]$$

and so the total complex for  $C(Z)$  is given by

$$C(Z) = \left[ \bigotimes_{e_i \in Z} R \xrightarrow{U_i} R \right] \otimes \left[ \bigotimes_{v \in W_2(Z)} R \xrightarrow{L(v)} R \right] \otimes \left[ \bigotimes_{v \in W_4(Z)} R \xrightarrow{Q(v)} R \right] \otimes \left[ \bigotimes_{v \in V_4(S)} R \xrightarrow{L(v)} R \right]$$

**Theorem 4.3.** *The filtered homology  $H(C(Z), d_0)$  is isomorphic to  $H_H(S - Z)$ .*

*Proof.* The  $U_i$  in the first tensor product form a regular sequence in  $R$ , so we can cancel all of these differentials. This has the effect of setting  $U_i$  equal to zero for all  $e_i$  in  $Z$ . Let  $R_Z$  be the quotient  $R/\{U_i = 0 \text{ for } e_i \in Z\}$ . Note that this is precisely the ground ring for the singular braid  $S - Z$ .

We are left with the complex

$$\left[ \bigotimes_{v \in W_2(Z)} R_Z \xrightarrow{L(v)} R_Z \right] \otimes \left[ \bigotimes_{v \in W_4(Z)} R_Z \xrightarrow{Q(v)} R_Z \right] \otimes \left[ \bigotimes_{v \in V_4(S)} R_Z \xrightarrow{L(v)} R_Z \right]$$

For each 4-valent vertex  $v$  in  $S - Z$ , we have tensor summands  $R_Z \xrightarrow{L(v)} R_Z$  and  $R_Z \xrightarrow{Q(v)} R_Z$ , which together give a summand of

$$\begin{array}{ccc} R(Z) & \xrightarrow{L(v)} & R(Z) \\ \downarrow Q(v) & & \downarrow Q(v) \\ R(Z) & \xrightarrow{L(v)} & R(Z) \end{array}$$

which is precisely the HOMFLY-PT summand  $C^{KR}(v)$ . For 2-valent vertices  $v$  in  $S - Z$ , there are two possibilities to consider -  $v$  is 2-valent in  $S$  ( $v \in W_2(Z)$ ), and  $v$  is 4-valent in  $S$  ( $v \in V_4(S), v \notin W_4(Z)$ ). When  $v$  is 2-valent in  $S$ , we get the summand

$$R_Z \xrightarrow{L(v)} R_Z$$

which is again the HOMFLY-PT summand  $C^{KR}(v)$  for  $S - Z$ . For  $v$  4-valent in  $S$ , let  $e_i$  and  $e_j$  be the outgoing edges at  $v$  and  $e_k$  and  $e_l$  the incoming edges at  $v$ . We since  $S - Z$  is 2-valent at  $v$ , we know that  $Z$  must include one outgoing edge and one incoming edge. Without loss of generality, assume they are  $e_i$  and  $e_k$ . To avoid confusion, we will write out the terms of the linear elements, as  $L(v)$  refers to  $U_i + U_j - U_k - U_l$  in  $S$ , while  $L(v)$  refers to  $U_j - U_l$  in  $S - Z$ .

In  $C(S)$ , we have the summand

$$R_Z \xrightarrow{U_i + U_j - U_k - U_l} R_Z$$

In the HOMFLY-PT complex for  $S - Z$ , on the other hand, we have the summand

$$R_Z \xrightarrow{U_j - U_l} R_Z$$

Fortunately, since  $e_i$  and  $e_k$  are in  $Z$ ,  $U_i$  and  $U_k$  are zero in  $R_Z$ , so  $U_i + U_j - U_k - U_l = U_j - U_l$ , making the above complexes isomorphic.

Thus, after canceling the Koszul complex on the edges in  $Z$ , we get exactly the HOMFLY-PT complex for  $S - Z$ . It follows that  $H(C(Z), d_0) \cong H^{KR}(S - Z)$

□

**Corollary 4.4.** *The filtered homology decomposes as the direct sum*

$$H(C(S), d_0) \cong \bigoplus_Z H^{KR}(S - Z)$$

**Remark 4.5. Signs.** Since we are discussing Koszul complexes, the  $\pm$  in the terms  $\pm(U_i - U_j)$  and  $\pm(U_i U_j - U_k U_l)$  are not relevant. Some will have to come with positive signs and some with negative to make  $d^2 = 0$ , but where they are doesn't impact the chain homotopy type. What we need to show is that the two bigons in each case come with *different* signs.

The two-valent vertex corresponds to a specific  $X$  marking in the diagram. This  $X$  lies within the same  $\alpha$  circle as  $O_i$  and the same  $\beta$  circle as  $O_j$ . In  $CFK^-$ , we do not allow discs to pass through the  $X$  basepoints. However, if we do allow them to pass through only this  $X$ , we get a new complex. In this complex,  $d^2$  is non-zero - instead, it is a multiple of the identity. This multiple is determined by the  $\alpha$  and  $\beta$  degenerations, which will correspond to the  $\alpha$  and  $\beta$  circles containing  $X$ . Since the  $\alpha$  circle contains  $O_i$ , it gives a coefficient of  $U_i$ , and similarly, the  $\beta$  circle gives a coefficient of  $U_j$ . The  $\alpha$  and  $\beta$  degenerations are known to come with opposite signs, so this gives

$$d^2 = \pm(U_i - U_j)I$$

Moreover, the additional differentials are also subject to the basepoint filtration, so we get

$$d_0^2 = \pm(U_i - U_j)I$$

This  $X$  basepoint lies inside a minimal bigon, and this bigon now contributes to the differential with a coefficient of  $\pm 1$ . The local contribution therefore must be

$$\begin{array}{ccc} & \pm(U_i - U_j) & \\ R & \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & R \\ & \pm 1 & \end{array}$$

which proves that the  $U_i$  and  $U_j$  come with opposite sign.

The argument for the quadratic term is the same, only instead of allowing discs to pass through an  $X$ , we are allowing them to pass through an  $XX$ . The  $\alpha$  degeneration is  $U_i U_j$  and the  $\beta$  degeneration is  $U_k U_l$ , and they must come with opposite sign, so we get the complex

$$\begin{array}{ccc} & \pm(U_i U_j - U_k U_l) & \\ R & \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & R \\ & \pm 1 & \end{array}$$

which proves that  $U_i U_j$  and  $U_k U_l$  come with opposite sign.

**4.4. Gradings.** The knot Floer complex comes equipped with two gradings: the Maslov grading  $M$  and the Alexander grading  $A$ . The differential decreases the Maslov grading by 1 and preserves the Alexander grading. Multiplication by  $U_i$  decreases the Maslov grading by 2 and decreases the Alexander grading by 1.

Certain linear combinations of the Maslov and Alexander gradings return analogs of the quantum and horizontal gradings from the Khovanov-Rozansky complex. Let  $gr_q$  be given by  $-2M + 2A$ , and  $gr_h$  by  $-2M + 4A$ . Note that the knot Floer differential has bigrading  $\{2, 2\}$  with respect to this differential and multiplication by  $U_i$  changes the bigrading by  $\{2, 0\}$ , the same as the Khovanov-Rozansky complex. Instead of the Maslov and Alexander gradings, we will henceforth use the quantum and horizontal gradings.

Before computing gradings, we need to introduce some terminology. Given a multi-cycle  $Z$ , let  $T_1(Z)$  denote the the number of vertices  $v \in V_4(S)$  at which  $Z$  contains the edges  $e_1$  and  $e_3$  in Figure 4. Similarly, let  $D_1(Z)$  denote the number of vertices at which  $Z$  contains the edges  $e_1$  and  $e_4$ ,  $D_2(Z)$  the number of vertices at which  $Z$  contains the edges  $e_2$  and  $e_3$ , and  $T_2(Z)$  denote the the number of vertices at which  $Z$  contains the edges  $e_2$  and  $e_4$ .

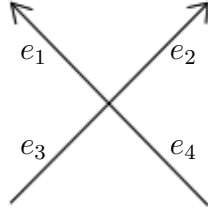


FIGURE 4. A labeled 4-valent vertex

We will now compute the bigrading on the knot Floer complex, up to an overall grading shift. Since the generators corresponding to the empty cycle give a complex which is isomorphic to the HOMFLY-PT complex for  $S$ , we can choose our overall shift so that this isomorphism preserves the bigrading.

Let  $Z$  denote a multi-cycle in  $S$  consisting of  $r(Z)$  cycles, and let  $x$  denote the generator corresponding to  $Z$  at the bottom of the Koszul complex (i.e. with the largest horizontal grading). Similarly, let  $y$  denote the generator corresponding to the empty cycle with the largest horizontal grading. In [10], Szabo and Ozsvath identify  $r(Z)$  differentials whose composition takes  $x$  to  $y$ , whose coefficient in  $R$  has degree  $T_2(Z) + \frac{1}{2}(D_1(Z) + D_2(Z))$ . Using the fact that differentials have bigrading  $\{2, 2\}$ , we can see that  $x$  and  $y$  differ in grading by

$$\{2r(Z) + 2T_2(Z) + D_1(Z) + D_2(Z), 2r(Z)\}$$

The bottom generator of the HOMFLY-PT complex has bigrading  $\{-|V_4(S)|, 0\}$ , so  $y$  does as well. Thus,  $x$  has bigrading

$$\{-|V_4(S)| - 2r(Z) + 2T_2(Z) + D_1(Z) + D_2(Z), -2r(Z)\}$$

The bottom generator of the HOMFLY-PT complex for  $S - Z$  has bigrading  $\{-|V_4(S - Z)|, 0\}$ , so we get the following graded version of Theorem 4.3:

$$H(C(Z), d_0) \cong H^{KR}(S-Z)\{|V_4(S-Z)| - |V_4(S)| - 2r(Z) + 2T_2(Z) + D_1(Z) + D_2(Z), -2r(Z)\}$$

The grading shift in this formula can be simplified somewhat. The quantity  $|V_4(S-Z)| - |V_4(S)|$  is the negative of the number of 4-valent vertices in  $S$  at which  $Z$  contains two edges.

$$|V_4(S-Z)| - |V_4(S)| = -T_1(Z) - T_2(Z) - D_1(Z) - D_2(Z)$$

Thus, the formula becomes

$$H(C(Z), d_0) \cong H^{KR}(S-Z)\{-2r(Z) + T_2(Z) - T_1(Z), -2r(Z)\}$$

and we get a graded version of Corollary 4.4:

$$H(C(S), d_0) \cong \bigoplus_Z H^{KR}(S-Z)\{-2r(Z) + T_2(Z) - T_1(Z), -2r(Z)\}$$

## 5. THE COMPOSITION PRODUCT

The composition product was first developed by Francois Jaeger in [5], where he showed that a certain sum of products of HOMFLY-PT polynomials corresponding to a link returned the HOMFLY-PT polynomial for that link. We will be interested in the specialization of this theory to singular diagrams and  $sl_n$  polynomials, developed by Wagner [12]. To discuss this version, we first need some definitions.

**Definition 5.1.** A subset  $E$  of the edges of  $S$  is called an *Eulerian circuit* if at each vertex  $v$  in  $S$ , the number of incoming edges to  $v$  in  $E$  is equal to the number of outgoing edges from  $v$  in  $E$ .

Note that this is a slight generalization of our previous definition of a multi-cycle, with the only difference being that Eulerian circuits are allowed to contain all of the edges at a 4-valent vertex in  $S$ .

**Definition 5.2.** Let  $f$  be a function from the edges in  $S$  to  $\{1, 2\}$ . We call  $f$  a *labeling* if  $f^{-1}(1)$  is an Eulerian circuit in  $S$ .

Observe that  $f^{-1}(1)$  is an Eulerian circuit if and only if  $f^{-1}(2)$  is an Eulerian circuit. Let  $S_{f,1}$  denote the singular braid obtained by deleting all edges from  $S$  labeled 2, and similarly, let  $S_{f,2}$  denote the singular braid obtained by deleting all edges from  $S$  labeled 1.

We can extend the definition of  $T_i$  and  $D_i$  to Eulerian circuits in the natural way - the vertices at which the circuit contains all four edges do not contribute to  $T_i$  nor  $D_i$ . With this extension, the composition product formula can then be stated as follows.

$$P_{m+n}(S) = \sum_{f \in L(S)} q^{\sigma_{m,n}(f)} P_m(S_{f,1}) P_n(S_{f,2})$$

where  $P_k(S)$  is the  $sl_k$  polynomial of  $S$ ,  $\sigma_{m,n}(f) = T_2(S_{f,1}) - T_1(S_{f,1}) - mr(S_{f,1}) + nr(S_{f,2})$ , and  $r(S_{f,i})$  is the number of strands in the singular braid  $S_{f,i}$ . Since our braid is oriented clockwise, our convention for  $r$  is the negative of the one in [12]. Wagner further showed that this relationship is true on the level of categorifications

$$H_{m+n}^{KR}(S) = \bigoplus_{f \in L(S)} H_m^{KR}(S_{f,1}) \otimes H_n^{KR}(S_{f,2}) \{\sigma_{m,n}(f)\}$$

Unfortunately, this isomorphism has been shown via an existence proof and no explicit construction exists for the standard definitions of  $H_n$ . We were, however, able to construct this isomorphism between Khovanov homology and the  $m = 1, n = 1$  case, which we describe in the final section.

## 6. THE SPECTRAL SEQUENCES

We are going to add differentials to the complex  $C(S)$  so that the total homology is isomorphic to  $H_{n+1}^{KR}(S)$  for any  $n \geq 1$ . These new differentials do not preserve the Alexander grading, so using the Alexander grading as a filtration, this induces a spectral sequence from  $H(C(S))$  to  $H_{n+1}^{KR}(S)$ .

The complex  $C(S)$  is constructed as a tensor product of complexes  $CFK^-(S)$  and a Koszul complex  $K(S)$  on linear elements.

$$C(S) = CFK^-(S) \otimes K(S)$$

The complex  $CFK^-(S)$  does not count discs which pass through the  $X$  or  $XX$  markings. For the new differential, we are going to count these discs with certain polynomial coefficients.

Each  $X$  marking in the Heegaard diagram corresponds to a 2-valent vertex  $v$  in  $S$ . Whenever a holomorphic disc passes through this  $X$  with multiplicity  $k$ , it picks up a coefficient of  $u_1(v)^k$ . The only exception is the special marking  $X_0$ , at which we still require discs to have multiplicity 0. Similarly, each  $XX$  corresponds to a 4-valent vertex  $v$  in  $S$ . If a holomorphic disc passes through this  $XX$  with multiplicity  $k$ , it picks up a coefficient of  $u_2(v)^k$ . We will call this new complex  $CFK_n^-(S)$ .

Note that there is no guarantee that the differential on this complex squares to zero - in fact, it doesn't.

To fix this, we will also modify the differential on the Koszul complex. Originally, it was given by

$$K(S) = \bigotimes_{v \in V_4(S)} R \xrightarrow{L(v)} R$$

We are going to add in differentials to make it a matrix factorization:

$$K_n(S) = \bigotimes_{v \in V_4(S)} R \begin{array}{c} \xrightarrow{L(v)} \\ \xleftarrow{u_1(v)} \end{array} R$$

The total complex  $C_n(S)$  is defined to be the tensor product of  $CFK_n^-(S)$  and  $K_n(S)$ .

$$C_n(S) = CFK_n^-(S) \otimes K_n(S)$$

**Lemma 6.1.** *The differential on  $C_n(S)$  satisfies  $d^2 = 0$ .*

*Proof.* At each vertex  $v$  in  $S$ ,  $d^2$  is going to have a local contribution of  $w_n(v)$ , which is given by

$$\sum_{e_i \in E_{out}} U_i^{n+1} - \sum_{e_j \in E_{in}} U_j^{n+1}$$

The lemma will then follow from the equality  $\sum_{v \in S} w_n(v) = 0$ .

The quantity  $d^2$  has two contributions, one from  $CFK_n^-(S)$  and one from  $K_n(S)$ . The contribution from  $K_n(S)$  can be computed directly to be

$$\sum_{v \in V_4(S)} L(v)u_1(v)$$

The contribution from  $CFK_n^-(S)$  can be computed via the  $\alpha$  and  $\beta$  degenerations. These degenerations always come with opposite sign, and without loss of generality we will assume the  $\alpha$  degenerations are positive and the  $\beta$  degenerations negative.

At each 2-valent vertex  $v$  in  $S$ , we have one  $\alpha$  circle and one  $\beta$  circle, shown in Figure 5. The  $\alpha$  circle contains  $U_i$  and  $X$ , and the  $X$  contributes coefficient  $u_1(v)$ , so the  $\alpha$  degeneration contributes  $U_i u_1(v)$ . Similarly, the  $\beta$  circle contains  $U_j$  and  $X$ , so its contribution is  $-U_j u_1(v)$ . Thus, the net contribution at  $v$  is  $(U_i - U_j)u_1(v)$ . This can be simplified to  $L(v)u_1(v) = w_n(v)$ .

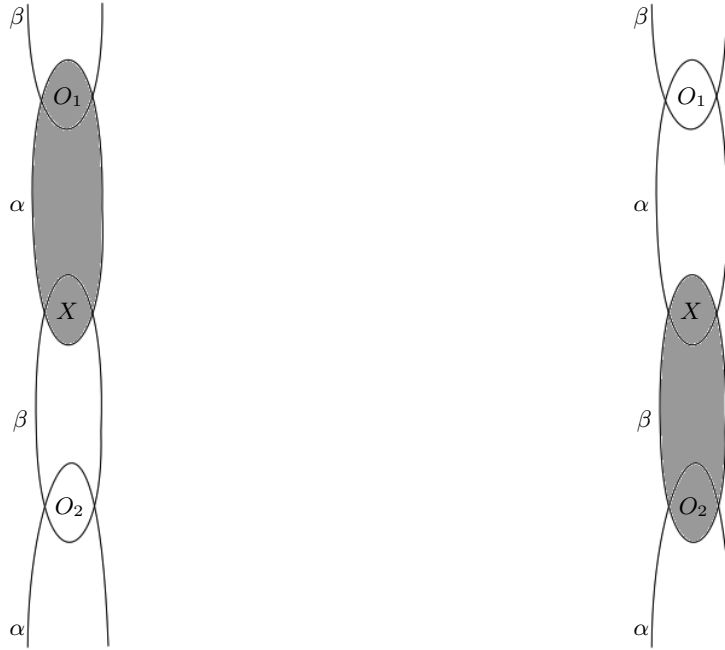
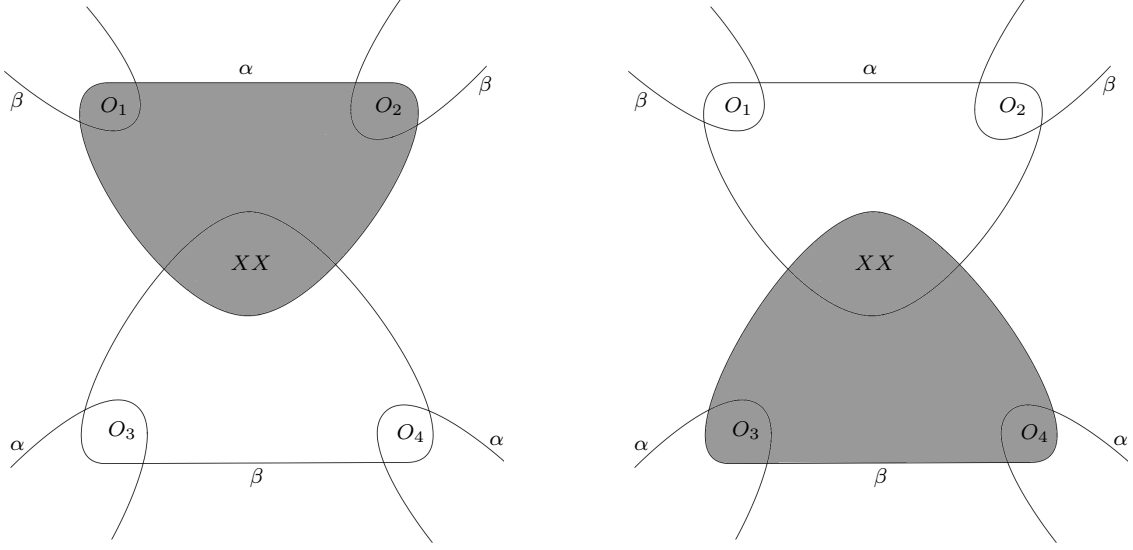


FIGURE 5.  $\alpha$ -degenerations (left) and  $\beta$ -degenerations (right) at a bivalent vertex

At each 4-valent vertex  $v$  in  $S$ , we also have one  $\alpha$  circle and one  $\beta$  circle, shown in Figure 6. The  $\alpha$  circle contains  $U_i, U_j$ , and  $X$ , and the  $X$  contributes coefficient  $u_2(v)$ , so its contribution is  $U_i U_j u_2(v)$ . The  $\beta$  circle contains  $U_k, U_l$ , and  $X$ , so its contribution is  $-U_k U_l u_2(v)$ . Thus, the net contribution at  $v$  is  $(U_i U_j - U_k U_l)u_2(v)$ . This can be simplified to  $Q(v)u_2(v)$ .

Thus, counting the contribution from  $K_n(S)$ , we see that

FIGURE 6.  $\alpha$ -degenerations (left) and  $\beta$ -degenerations (right) at a four-valent vertex

$$\begin{aligned}
d^2 &= \sum_{v \in V_2(S)} w_n(v) + \sum_{v \in V_4(S)} Q(v)u_2(v) + \sum_{v \in V_4(S)} L(v)u_1(v) \\
&= \sum_{v \in V_2(S)} w_n(v) + \sum_{v \in V_4(S)} L(v)u_1(v) + Q(v)u_2(v) \\
&= \sum_{v \in V_2(S)} w_n(v) + \sum_{v \in V_4(S)} w_n(v) \\
&= \sum_{v \in S} w_n(v) = 0
\end{aligned}$$

□

We can extend the basepoint filtration from Section 4.2 to make  $C_n(S)$  a filtered complex - since we still require discs to have multiplicity 0 at  $X_0$ , the same argument works as in the proof of Lemma 4.1. Let  $d_i$  denote the differentials which change the filtration level by  $i$ . As before,  $d_0$  must preserve multi-cycles, so the homology  $H(C_n(S), d_0)$  splits over the multi-cycles

$$H(C_n(S), d_0) = \bigoplus_Z H(C_n(Z), d_0)$$

We want to compute the complex  $C_n(Z)$ . Recall that  $C(Z)$  was computed to be

$$\left[ \bigotimes_{e_i \in Z} R \xrightarrow{U_i} R \right] \otimes \left[ \bigotimes_{v \in W_2(Z)} R \xrightarrow{L(v)} R \right] \otimes \left[ \bigotimes_{v \in W_4(Z)} R \xrightarrow{Q(v)} R \right] \otimes \left[ \bigotimes_{v \in V_4(S)} R \xrightarrow{L(v)} R \right]$$

We can therefore compute  $C_n(Z)$  by adding the new differentials to this complex. It is not hard to see that the only new discs in  $CFK_n^-(Z)$  correspond to bigons containing  $X$  or  $XX$  basepoints. For example, let  $e_i$  be an edge in  $Z$ , with  $x$  and  $y$  the two intersection points corresponding to  $e_i$ . When we weren't allowing discs to pass through  $X$  or  $XX$ , the only



disc connecting  $x$  and  $y$  was the bigon containing  $U_i$ . This contributed the tensor summand of

$$R \xrightarrow{U_i} R$$

However, when we allow discs to pass through  $X$  and  $XX$ , we get two new bigons which map from  $y$  to  $x$ , shown in Figure 7.



FIGURE 7. Two new bigons from  $y$  to  $x$

The type of contribution from these bigons depends on whether the endpoint vertices of  $e_i$  are 2-valent or 4-valent, which is why we only showed the local portion of the bigons in Figure 7. However, in either case the contribution has a coefficient of degree  $n$ . We will denote this coefficient by  $p(e_i)$  (the precise polynomial will not be relevant for our computations). The tensor summand then becomes

$$R \xrightleftharpoons[p(e_i)]{U_i} R$$

For a vertex  $v$  in  $W_2(Z)$ , there are two intersection points  $x$  and  $y$  corresponding to  $v$ . In  $C(Z)$ , they contributed a tensor summand of

$$R \xrightarrow{L(v)} R$$

In  $C_n(Z)$ , we have an extra differential corresponding to the bigon from  $y$  to  $x$  through  $X$  (See Figure 8a). Since  $X$  carries a coefficient of  $u_1(v)$ , the summand becomes

$$R \xrightleftharpoons[u_1(v)]{L(v)} R$$

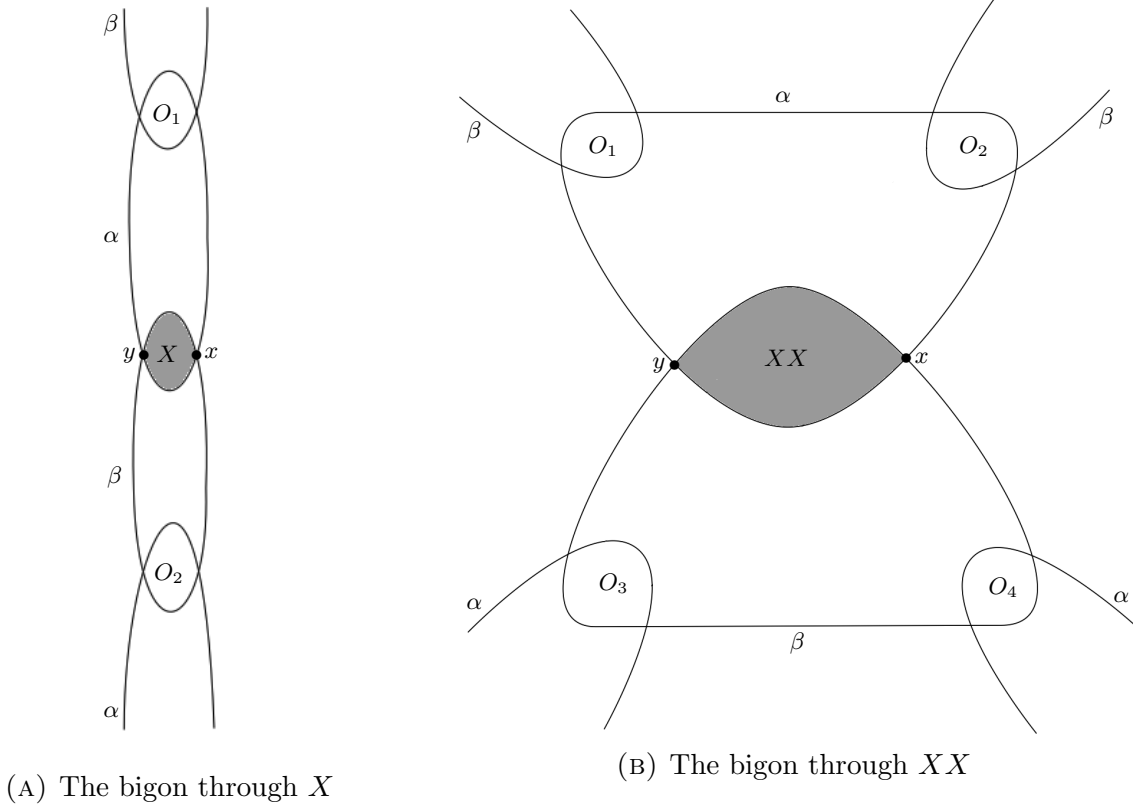
Similarly, for 4-valent vertices  $v$  in  $W_4(Z)$ ,  $C(Z)$  contains a tensor summand

$$R \xrightarrow{Q(v)} R$$

In  $C_n(Z)$ , we have an extra differential corresponding to the bigon through  $XX$  shown in Figure 8b. The  $XX$  contributes a coefficient of  $u_2(v)$ , so the summand becomes

$$R \xrightleftharpoons[u_2(v)]{Q(v)} R$$

Finally, the complex  $K(S)$  gets changed to  $K_n(S)$ , so the whole complex for  $C_n(Z)$  can be written as

FIGURE 8. New differentials passing through  $X$  and  $XX$ 

$$\left[ \bigotimes_{e_i \in Z} R \xrightleftharpoons[p(e_i)]{U_i} R \right] \otimes \left[ \bigotimes_{v \in W_2(Z)} R \xrightleftharpoons[u_1(v)]{L(v)} R \right] \otimes \left[ \bigotimes_{v \in W_4(Z)} R \xrightleftharpoons[u_2(v)]{Q(v)} R \right] \otimes \left[ \bigotimes_{v \in V_4(S)} R \xrightleftharpoons[u_1(v)]{L(v)} R \right]$$

Now that we have our complex computed, we want to compare its homology with  $H_n^{KR}(S)$ . We will denote the differentials which do not pass through any  $X$  or  $XX$  basepoints  $d_{0+}$ , and the new differentials  $d_{0-}$ . Observe that with respect to the bigrading  $(gr_q, gr_h)$  introduced in Section 4.4,  $d_{0+}$  has bigrading  $\{2, 2\}$ , and  $d_{0-}$  has bigrading  $\{2n, -2\}$ .

**Lemma 6.2.** *Up to an overall grading shift, the complex  $(H(C'_n(Z), d_{0+}), d_{0-}^*)$  is isomorphic to  $(H(C_n(S - Z), d_+), d_-)$*

*Proof.* It follows from Theorem 4.3 that  $H(C_n(Z), d_{0+}) \cong H^{KR}(S - Z)$ . To complete the proof, we need to show that  $d_{0-}^*$  corresponds to the  $d_-$  differential under this isomorphism. For vertices which are 2-valent in both  $S$  and  $S - Z$  (i.e.  $v \in W_2(Z)$ ), this is obvious. The same is true for vertices which are 4-valent in both  $S$  and  $S - Z$  (i.e.  $v \in W_4(Z)$ ).

The only identification which is non-trivial is that the  $d_{0-}$  differential corresponding to a vertex which is 4-valent in  $S$  but 2-valent in  $S - Z$  is the same as the  $d_-$  differential on the 2-valent vertex in  $S - Z$ . Let  $e_i, e_j$  be the outgoing edges at  $v$  and  $e_k, e_l$  the incoming edges. The multi-cycle  $Z$  must contain one incoming and one outgoing edge - without loss of generality, assume  $Z$  contains  $e_i$  and  $e_k$ . The coefficient of the  $d_-$  differential is given by

$$\frac{U_j^{n+1} - U_l^{n+1}}{U_j - U_l}$$

while the coefficient of the  $d_{0-}$  differential is given by

$$\frac{U_i^{n+1} + U_j^{n+1} - U_k^{n+1} - U_l^{n+1} - Q(v)u_2(v)}{U_i + U_j - U_k - U_l}$$

Recall that to achieve the isomorphism in Theorem 4.3, we first cancelled the Koszul complex on the  $U_i$  for  $e_i$  in  $Z$ , as these elements formed a regular sequence. We therefore want to show that these two coefficients are equal in  $R_Z = R/\{U_i = 0 \text{ for } e_i \in Z\}$ . Substituting  $U_i = U_k = 0$  into the above equation and noting that this causes  $Q(v)$  to be zero, we get the desired equality.  $\square$

Define  $H^\pm(C_n(S)) = H(H(C_n(S), d_{0+}), d_{0-}^*)$ . Since both  $d_{0+}$  and  $d_{0-}$  are homogeneous with respect to the bigrading, this homology is bigraded as well. Applying the lemma and adding in the gradings from Section 4.4, we see that

$$H^\pm(C_n(S)) \cong \bigoplus_Z H^\pm(C_n^{KR}(S - Z))\{-2r(Z) + T_2(Z) - T_1(Z), -2r(Z)\}$$

Recall from Lemma 3.3 that  $H^\pm(C_n^{KR}(S - Z))$  lies in a single horizontal grading, namely  $-2r(S - Z)$ . Adding in the shift, the homology corresponding to a multi-cycle  $Z$  must lie in horizontal grading  $-2r(S - Z) - 2r(Z) = -2r(S)$ . But this does not depend on  $Z$ , so we have shown the following:

**Lemma 6.3.** *The homology  $H^\pm(C_n(S))$  lies in a single horizontal grading.*

The original differentials on  $C(S)$  all have bigrading  $\{2, 2\}$ . The new differentials on  $CFK_n^-(S)$  have bigrading  $\{2 + 2k(n - 1), 2 - 4k\}$ , where  $k$  is the sum of the multiplicities of the holomorphic discs at all  $X$  and  $XX$  markings. The new differentials on  $K_n(S)$  all have bigrading  $\{2n, -2\}$ . Thus, all differentials on  $C_n(S)$  the horizontal grading by 2 (mod 4). This implies that no induced differentials can have horizontal grading 0, which tells us that the remaining differentials on our complex are all trivial, giving us the following:

**Lemma 6.4.** *The total homology  $H(C_n(S), d)$  is isomorphic to  $H^\pm(C_n(S))$ .*

This isomorphism is singly-graded with grading  $gr_n = gr_q + (n - 1)gr_h/2$ , as the total differential on  $C_n(S)$  is homogeneous of degree  $n + 1$  with respect to this grading.

Putting all of this together, we get an explicit formula for  $H(C_n(S), d)$ .

$$H(C_n(S), d) \cong \bigoplus_Z H_n^{KR}(S - Z)\{-(n + 1)r(Z) + T_2(Z) - T_1(Z)\}$$

This homology is beginning to resemble the composition product formula - to make this relationship concrete, we need the following result from  $sl_1$  homology.

**Lemma 6.5.** *Let  $f$  be a labeling of  $S$ . The  $sl_1$  homology of  $S_{f,1}$  is given by*

$$H_1^{KR}(S_{f,1}) = \begin{cases} \mathbb{Q}\{0\} & \text{if } S_{f,1} \text{ is a multi-cycle} \\ 0 & \text{otherwise} \end{cases}$$

In other words, whenever  $S_{f,1}$  contains all four edges at a 4-valent vertex  $v$ ,  $H_1(S_{f,1})$  is trivial. This follows from the fact that when  $n = 1$ ,  $u_2 = 2$ , which is a unit in  $R$ . Thus, our homology can be expressed as

$$H(C_n(S), d) \cong \bigoplus_f H_1^{KR}(S_{f,1}) \otimes H_n^{KR}(S_{f,2}) \{-(n+1)r(S_{f,1}) + T_2(S_{f,1}) - T_1(S_{f,1})\}$$

Giving this complex an overall grading shift of  $nr(S)$ , we get

$$H(C_n(S), d) \cong \bigoplus_f H_1^{KR}(S_{f,1}) \otimes H_n^{KR}(S_{f,2}) \{T_2(S_{f,1}) - T_1(S_{f,1}) - r(S_{f,1}) + nr(S_{f,2})\}$$

But this is exactly the composition product formula  $(1, n)$ , so we have proved the following:

**Theorem 6.6.** *The total homology  $H(C_n(S), d)$  is isomorphic to  $H_{n+1}^{KR}(S)$ .*

**Corollary 6.7.** *For all  $n \geq 1$ , there is a spectral sequence whose  $E_1$  page is  $H(C(S))$  which converges to  $H_{n+1}^{KR}(S)$ .*

*Proof.* All of the original differentials on  $C(S)$  have Alexander grading 0. The new differentials on  $CFK_n^-$  have Alexander grading  $k(-n-1)$ , where  $k$  is the sum of the multiplicities of the disc at the  $X$  and  $XX$  basepoints, and the new differentials on the Koszul complex have Alexander grading  $-n-1$ . In particular, all of the new differentials strictly decrease the Alexander grading, so it induces a filtration with respect to which the filtered homology is  $H(C(S))$ . Thus, the corresponding spectral sequence has  $E_1$  page  $H(C(S))$ , and converges to the total homology  $H(C_n(S)) \cong H_{n+1}^{KR}(S)$ . □

In general, while there is an isomorphism  $H(C_n(S)) \cong H_{n+1}^{KR}(S)$ , it can not currently be made explicit. The exception is for  $n = 1$ , where we identify an explicit isomorphism between  $H(C_1(S))$  and  $H_2^{KR}(S)$ , or Khovanov homology.

## 7. A CANONICAL ISOMORPHISM WITH KHOVANOV HOMOLOGY

For the  $m = 1, n = 1$  version of the composition product, we identify a canonical isomorphism with the Khovanov homology of the diagram obtained by replacing singularizations with unoriented smoothings (this is how the isomorphism between Khovanov homology and  $H_2^{KR}$  is obtained, see [4]). The composition product formula specializes to

$$H_2^{KR}(S) \cong \bigoplus_f H_1^{KR}(S_{f,1}) \otimes H_1^{KR}(S_{f,2}) \{T_2(S_{f,1}) - T_1(S_{f,1}) - r(S_{f,1}) + r(S_{f,2})\}$$

Given a singular braid  $S$ , let  $Smooth(S)$  denote the diagram in which each 4-valent vertex has been replaced with the unoriented smoothing. The diagram  $Smooth(S)$  is a union of embedded circles in the plane. Let  $A$  denote a subset of the edges of  $S$  corresponding to one of the circles.

Since  $H_1^{KR}(S_{f,i})$  is trivial whenever  $S_{f,i}$  contains all 4-valent vertices, the contribution from a labeling  $f$  will be trivial unless both  $S_{f,1}$  and  $S_{f,2}$  contain exactly one incoming edge and one outgoing edge at each 4-valent vertex.

**Lemma 7.1.** *Suppose the labeling  $f$  has non-trivial contribution. Then once  $f$  has assigned a value to an edge in  $A$ , the rest of the assignments to edges in  $A$  are uniquely determined.*

*Proof.* Suppose  $e_i \in A$ , and without loss of generality, suppose  $f(e_i) = 1$ . We will traverse the edges in  $A$  clockwise. Each time we come to a 2-valent vertex in  $S$ , we know that the next edge must have the same labeling as the previous one. Every time we come to a 4-valent vertex in  $S$  (which has been smoothed in  $Smooth(S)$ ), the next edge in  $A$  must have the other labeling.

□

For example, in Figure 9, there are three circles in  $Smooth(S)$ :  $e_1e_6e_7e_8$ ,  $e_2e_3$ , and  $e_4e_5$ . If a labeling  $f$  has non-trivial contribution, then  $f(e_1) = 1$  implies  $f(e_6) = 2$ ,  $f(e_7) = 1$ , and  $f(e_8) = 2$ . (If  $f(e_1) = 2$ , the rest of the assignments are toggled as well.) Similarly, if  $f(e_2) = 1$  then  $f(e_3) = 2$ , and if  $f(e_4) = 1$  then  $f(e_5) = 2$ .

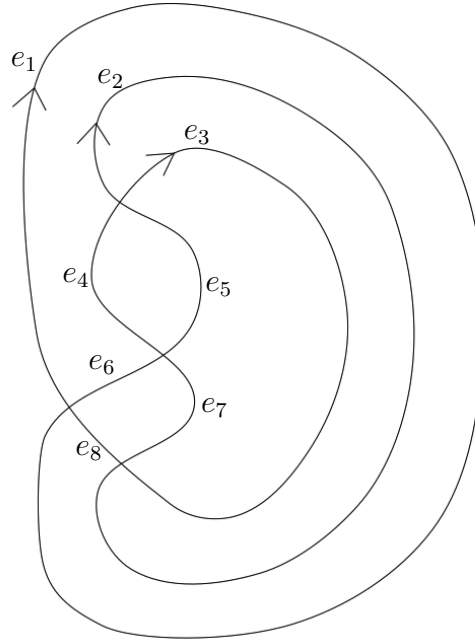


FIGURE 9. A Singular Braid Diagram

Since the edge  $e$  has 2 possible labelings, there are  $2^k$  labelings  $f$  with non-trivial contribution, where  $k$  is the number of circles in  $Smooth(S)$ . Each of these contributes

$$H_1^{KR}(S_{f,1}) \otimes H_1^{KR}(S_{f,2}) \cong \mathbb{Q}$$

to the homology, so  $H(C_1(S))$  is  $2^k$ -dimensional.

Let  $A_k$  be the set of edges in  $A$  which lie on the  $k$ th strand of  $S$ , with the 1st strand being the outermost. Given a labeling  $f$  with non-trivial contribution, the above argument shows

also that all edges in  $A_k$  must have the same labeling, and that the edges in  $A_{k+1}$  have the opposite labeling of those in  $A_k$ . Given a circle  $A$ , let  $\min(A)$  be the minimal  $k$  such that  $A_k$  is non-empty. Let  $1_A$  denote the local generator of  $H(C_1(S))$  which assigns 1 to  $A_{\min(A)}$ , and  $x_A$  the local generator which assigns 2 to  $A_{\min(A)}$ .

**Lemma 7.2.** *If  $gr(1_A)$  and  $gr(x_A)$  are the gradings of the two generators, then  $gr(x_A) = gr(1_A) + 2$ .*

*Proof.* There are two cases to consider - the one where  $A$  contains the centerpoint of the braid, and the one where it does not. Let  $v$  be a 4-valent vertex in  $S$  which connects an edge in  $A_k$  to an edge in  $A_{k+1}$ . When  $k = \min(A) \pmod{2}$ , the switch from  $1_A$  to  $x_A$  increases the contribution of  $T_2(S_{f,1}) - T_1(S_{f,1})$  by 1, regardless of what  $f$  assigns to the other two edges at  $v$ . Similarly, when  $k = \min(A) + 1 \pmod{2}$ , the switch from  $1_A$  to  $x_A$  decreases the contribution of  $T_2(S_{f,1}) - T_1(S_{f,1})$  by 1.

When  $A$  contains the centerpoint of the braid, the number of vertices for which  $k = \min(A) \pmod{2}$  is equal to the number of vertices for which  $k = \min(A) + 1 \pmod{2}$ , but  $r(S_{f,1})$  is one higher for  $1_A$  than for  $x_A$ . Thus the difference in grading shifts from the formula

$$\{T_2(S_{f,1}) - T_1(S_{f,1}) - r(S_{f,1}) + r(S_{f,2})\}$$

is 2, with the difference coming from the  $r(S_{f,i})$  terms.

When  $A$  does not contain the centerpoint of the braid, the number of vertices for which  $k = \min(A) \pmod{2}$  is 2 more than the number of vertices for which  $k = \min(A) + 1 \pmod{2}$ , while the values of  $r(S_{f,i})$  remain unchanged. Thus the difference in grading shifts is still 2, but with the difference coming from the  $T_2(S_{f,1}) - T_1(S_{f,1})$  term.  $\square$

Let  $\mathcal{A}$  denote the graded algebra  $\mathbb{Q}[x]/x^2 = 0$ , where  $x$  has grading 2. We have shown that, up to an overall grading shift,

$$C_1(S) \cong \mathcal{A}^{\otimes n}$$

where  $n$  is the number of circles in  $\text{Smooth}(S)$ .

*Remark 7.3.* It can be shown that the module structure of  $C_1(S)$  also aligns with the module structure on Khovanov homology, but this requires a somewhat lengthier argument that we choose not to include in this paper. It is also interesting to note that the grading  $gr_n$  is  $2(A - M)$ , which is twice the  $\delta$ -grading on knot Floer homology. Since the conjectured spectral sequence from Khovanov homology to  $HFK$  would converge to  $\delta$ -graded  $HFK$ , it seems possible that an exact triangle for our construction could give progress on this conjecture.

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